Exact scalings in competitive growth models

L. A. Braunstein\textsuperscript{1,3} and Chi-Hang Lam\textsuperscript{2}

\textsuperscript{1}Departamento de Física, Facultad de Ciencias Exactas y Naturales
Universidad Nacional de Mar del Plata
Funes 3350, 7600 Mar del Plata, Argentina

\textsuperscript{3}Center for Polymer Studies and Department of Physics
Boston University, Boston, MA 02215, USA

\textsuperscript{2}Department of Applied Physics, Hong Kong
Polytechnic University, Hung Hom, Hong Kong

Abstract

A competitive growth model (CGM) describes aggregation of a single type of particle under two distinct growth rules with occurrence probabilities $p$ and $1 - p$. We explain the origin of scaling behaviors of the resulting surface roughness with respect to $p$ for two CGMs which describe random deposition (RD) competing with ballistic deposition (BD) and RD competing with the Edward Wilkinson (EW) growth rule. Exact scaling exponents are derived and are in agreement with previously conjectured values [1, 2]. Using this analytical result we are able to derive theoretically the scaling behaviors of the coefficients of the continuous equations that describe their universality classes. We also suggest that, in some CGM, the $p$–dependence on the coefficients of the continuous equation that represents the universality class can be non trivial. In some cases the process cannot be represented by a unique universality class. In order to show this we introduce a CGM describing RD competing with a constrained EW (CEW) model. This CGM show a transition in the scaling exponents from RD to a Kardar-Parisi-Zhang behavior when $p \to 0$ and to a Edward Wilkinson one when $p \to 1$. Our simulation results are in excellent agreement with the analytic predictions.

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I. INTRODUCTION

Evolving interfaces or surfaces are of great interest due to their potential technological applications. These interfaces can be found in many physical, chemical and biological processes. Examples include film growth either by vapour deposition or chemical deposition [3], bacterial colony growth [4] and propagation of forest fire [6].

For a system exhibiting dynamical scaling, the r.m.s. roughness $W$ of an interface is characterized by the following scaling with respect to time $t$ and the lateral system width $L$:

$$W(L,t) \sim L^\alpha f(t/L^\beta),$$

where the scaling function $f(u)$ behaves as $f(u) \sim u^\beta$ with $\beta = z/\alpha$ for $u \ll 1$ and $f(u) \sim$ constant for $u \gg 1$. The exponent $\alpha$ is the roughness exponent that describes the dependence of the saturated surface roughness versus the lateral system size, while the exponent $\beta$ describes scaling at an early stage when finite-size effects are negligible. The crossover time between the two regimes is $t_s = L^\beta$.

A widely studied phenomenological equation representing the non-equilibrium growth of such interfaces is the Kardar-Parisi-Zhang (KPZ) equation. In $1 + 1$ dimensions, it states that:

$$\frac{\partial h(x,t)}{\partial t} = \nu_0 \frac{\partial^2 h(x,t)}{\partial x^2} + \lambda_0 \left( \frac{\partial h(x,t)}{\partial x} \right)^2 + \eta(x,t)$$  \hspace{1cm} (1)

where $h(x,t)$ is the local surface height at lateral coordinate $x$ and time $t$. The coefficients $\nu$ and $\lambda$ represent the strength of the linear and nonlinear surface smoothing terms respectively. The noise $\eta(x,t)$ is Gaussian with zero variance and covariance

$$\langle \eta(x,t) \eta(x',t') \rangle = 2D_0 \delta(x-x') \delta(t-t')$$

where $D_0$ is the strength of the noise. The exponents characterizing the KPZ equation in the highly nonlinear strong coupling are $\alpha = 1/2$ and $\beta = 1/3$. In contrast, at $\lambda = 0$ the linear Edward Wilkinson (EW) equation is recovered leading to the weak coupling exponents $\alpha = 1/2$ and $\beta = 1/4$. When both $\nu$ and $\lambda$ are zero the growth reduces to simple random deposition (RD) with $\beta = 1/2$ but a lack of any saturation regime.

There has been a recent interest in the study of competitive growth models (CGM) analyzing the interplay and competition between two current growth processes for a single surface. These CGM are often more realistic in describing growing in real materials [5],
in which more than one microscopic growth mode usually exist. For example, two distinct growth phases were observed in experiments on interfacial roughening in Hele-Shaw flows [7, 8] as well as in simulations on electrophoretic deposition of polymer chains [9, 10]. The resulting universalities from these competing processes are not well understood [1, 2, 11, 12]. Recently Horowitz et al. [2] introduced a CGM called BD/RD in which the microscopic growth rule follows either that of the ballistic deposition (BD) model with probability $p$ or simple random deposition (RD) with probability $1 - p$. This system exhibits a transition at a characteristic time from RD to KPZ. They found numerically that the scaling behavior of $W$ in that model is giving by the empirical form

$$W \sim \frac{L^\alpha}{p^\delta} \left( \frac{t}{p^{-\nu} L^z} \right)$$

(2)

Based on numerical estimates, exact values

$$\delta = 1/2 \quad \text{and} \quad y = 1$$

(3)

have been conjectured for the BD/RD model. Based on this conjecture, the authors concluded using scaling arguments that the model follows Eq. (1) with $\nu \sim p$ and $\lambda \sim p^{3/2}$. Subsequently, a similar CGM namely EW/RD describing the competition between the EW model and RD [1]. The simulations showed that Eq. (2) also holds with a different set of exponents which are conjectured to be

$$\delta = 1 \quad \text{and} \quad y = 2$$

(4)

This model can also be described by Eq. (1) with $\nu \sim p^2$ and $\lambda = 0$ [1, 11].

In this paper, we explain the scaling form (2) and derive rigorously the exact exponents $\delta$ and $y$ using simple arguments. In addition, from the above examples of CGM, one might be tempted to conclude that a CGM based on RD and a model in the KPZ (EW) universality class should always lead to an overall process in the KPZ (EW) class. We suggest that these naive predictions of the universality is not always correct. Close examination of the microscopic details of the growth models is indeed essentially. This is illustrated by introducing a constrained EW (CEW) model. Although this model essentially belongs to the EW class, a CGM in the form CEW/RD at sufficiently small $p$ results in an overall process in the KPZ universality class. It also demonstrates that a CGM can crossover from one universality class to another by varying $p$. 

3
II. EXACT SCALINGS FOR CGMS

In the RD model a particle is dropped at a randomly selected column increasing the local surface height by one. For the CGMs BD/RD or EW/RD described above, a particle is deposited on the surface following a RD process with probability $1 - p$ and by another process $A$ (which is either BD or EW) with probability $p$. Now, we derived analytically the exact exponents $\delta$ and $y$ which characterize the $p$ dependence of the scaling behavior of $W$ given in Eq. (2). We consider $p \to 0$. At each unit time, $L$ particles are deposited. The average time interval between any two consecutive $A$ events at any column $i$ is $\tau = 1/p$. During this period, $\tau - 1 \simeq \tau$ atoms on average are directly stacked onto the surface according to the simple RD rule. The local height at column $i$ hence increases by $\eta_i$ which is an independent Gaussian variable with mean $\overline{\eta} = \tau$ and standard deviation $\sigma_\eta = \sqrt{\tau}$ according to the central limit theorem. The mean however only leads to an irrelevant rigid shift of the whole surface. We can easily apply a vertical translation so that $\overline{\eta} = 0$. After these $\tau$ RD events at column $i$, one $A$ event on average is expected at the same column.

Now we consider $A$ to be the BD process. The CGM is then the BD/RD model. When a BD event occurs at column $i$, its height is updated in the simulations according to $h_i \rightarrow \max\{h_{i-1}, h_{i+1}, h_i + 1\}$. In the limit $p \to 0$ so that $\sigma_\eta \gg 1$, the height of the atom is negligible compared with the increments due to the RD events. The growth rule hence reduces to

$$h_i \rightarrow \max\{h_{i-1}, h_{i+1}, h_i\}$$ (5)

We have now arrived at a limiting BD/RD model defined as follow: At every coarsened time step $\tau = 1/p$, the local height $h_i$ at every column $i$ first changes by an additive Gaussian noise term $\eta_i$ with mean zero and standard deviation $\sigma_\eta = \sqrt{\tau}$. Then the limiting BD growth rule in Eq. (5) is applied to every column $i$. A more careful analysis should account for the fact that the BD events at various columns indeed occur randomly and asynchronously during the period $\tau$ but this will not affect our result. In this limiting model, the time and the vertical length scales are determined completely by $\tau$ and $\sigma_\eta$ respectively. Therefore, time scales as $t \sim \tau = 1/p$ while roughness scales as $W \sim \sigma_\eta \sim 1/p^{1/2}$. This explains the $p$ dependence of the scaling form in Eq. (2). In particular, we obtain the exact exponents $y = 1$ and $\delta = 1/2$ in agreement with values in Eq. (3) first conjectured in Ref. [1] but not derived analytically before.
Next, we assume that $A$ represents the EW growth rule instead. For this growth rule, a particle is dropped at a random column but when it reaches the surface it is allowed to relax to the lower of the nearest neighboring columns. If the heights at both nearest neighbors are lower than the selected one the relaxation is directed to either of them with equal probability. Our CGM now becomes a EW/RD model. The corresponding derivation of the characteristic length and time scales is similar to that for the BD/RD case. Assuming again $p \to 0$, the average time interval between any two consecutive EW events at any given site is $\tau = 1/p$. Consider a characteristic time $n \tau$. On average $n \tau$ RD events occur at any given site leading height increments with a standard deviation $\sqrt{n \tau}$. However, only $n$ EW events take place. The resulting smoothing dynamics is such that a big step for instance will typically decrease in height by $n$. For scaling to hold, the two length scales $\sqrt{n \tau}$ and $n$ have to be identical and we obtain $\tau \sim n$. The characteristic time scale considered is hence $n \tau \sim 1/p^2$, while a characteristic length scale for the surface height is $\sqrt{n \tau} \sim 1/p$. We have hence derived $y = 2$ and $\delta = 1$ previously conjectured in Ref. [1].

III. CEW AND CEW/RD MODELS

We now introduce the constrained EW model (CEW) which is a generalization of the EW model. In $1 + 1$ dimensions, particles are aggregated by the following rules. We choose a site $i$ at random among the $L$ possible sites. The surface height $h_i$ at the selected column is increased by one if this height is lower than the values $h_{i\pm1}$ at the neighboring columns. Otherwise either $h_{i-1}$ or $h_{i+1}$, whichever smaller, is increased to

$$h_{i\pm1} = \max\{h_{i\pm1} + 1, h_i - c\}$$

If $h_{i-1} = h_{i+1}$, either one will be updated with equal probability. Growth at $i \pm 1$ physically represents the rollover of a newly dropped particle to a lower site nearby under the influence of gravity for example. In the original EW model, there is no limit in the vertical distance transversed during the rollover and the particle can in principle slides down a very deep cliff if one exists next to column $i$. This is unphysical if there is a finite chance for the sideway sticking of the particle to the cliff. In the CEW growth rule defined in Eq. (6), this vertical drop during rollover is limited to $c$ by the process of sideway sticking. As $c \to \infty$, it is easy to see that CEW reduces to the standard EW model. At $c = 0$, sideway sticking of
particles occurs frequently and CEW behaves similarly to BD, although the precise growth rules are different. In the rest of this paper, we put $c = 4$. As will be demonstrated by the simulations presented later, the CEW model at $c = 4$ at practical length and time scales belongs to the EW universality class. Finally, we can define the CEW/RD model which is a CGM based on RD and CEW. Similar to the definitions of other CGM defined above, at each simulation step, the CEW growth rule is applied with probability $p$ while a RD event occurs with probability $1 - p$. Time $t$ is then increased by $1/L$.

Now we present the simulation results. In Fig. 1(a) and Fig. 1(b) we show the Log-Log plot of $W/L^\alpha$ as function of $t/L^z$ for two limiting values of $p$. For $p \to 0$ the behavior is consistent with the KPZ universality class with $\alpha = 1/2$ and $z = 3/2$, while for $p \to 1$ the system behaves as predicted by the EW equation with $\alpha = 1/2$ and $z = 2$. The initial regime corresponding to the RD deposition, with $\beta = 1/2$ does not scale with the system size. In order to show that the universality class depends on $p$, we compute $\beta$ as function of $t$ using successive slopes defined in Ref [13].

Figure 2 plots $\beta$ as a function of log $t$ for different values of $p$. At the beginning $\beta = 1/2$ as expected for the initial RD regime. After this early regime, the system evolves either to
FIG. 2: Plot of the dynamic exponent $\beta$ as function of $\log t$ for $L = 8192$ and different $p$ values: $p = 0.02$ ($\circ$), $p = 0.04$ ($\square$), $p = 0.08$ ($\diamond$), $p = 0.16$ ($\triangle$), $p = 0.32$ ($+$), $p = 0.64$ ($\ast$), and $p = 1.0$ ($\times$) showing the change in the behavior of $\beta$ with time. The $\beta$ values were computed over 100 realizations. The arrows are used as guides to show the asymptotic exponents

the KPZ class with $\beta = 1/3$ for $p \to 0$ or to the EW class for $p \to 1$ with $\beta = 1/4$. For intermediate $p$ values (after the RD regime), the system behaves as in the weak coupling of Eq. (1). It is easy to observe a transition from an EW to a KPZ for $p \geq 0.32$ while for $p \to 0$ the system always belongs to the KPZ universality class.

IV. GENERALIZED CONTINUOUS EQUATIONS AND SCALING

As we show above the CEW/RD model has a transition from a KPZ to a EW as the tuning parameter $p$ goes from 0 to 1. Thus the nonlinear coefficient $\lambda$ of the KPZ equation has to vanish as $p \to 1$. In order to understand the functional form of $\lambda(p)$ we perform a finite size scaling analysis of the growth velocity based on [14]:

$$\Delta v(L) \sim \lambda L^{-\alpha} \quad \text{for } t \gg t_s \quad (7)$$

where $\Delta v(L, t) = v(L = 1024, t) - v(L = 10, t)$ and $v(L, t) = \langle dh/dt \rangle$. The $\Delta v$ correction should go to zero when the nonlinear term $\lambda$ vanishes. Thus, using Eq. (7) we can determine how $\lambda$ in the KPZ equation depends on $p$. In Fig. 3 we plot $\Delta v$ as function of $p$ for fixed $L$ to show the $p$ dependence of $\lambda$. From the plot we can see that the functional form for the
FIG. 3: Linear plot of \( \lambda \) as function of \( p \). The averages where taken over typically 500 realizations. The errors bars were computed using 5 independent sets of 500 realizations each.

CEW/RD CGM is totally different from the scaling form \( \lambda(p) \sim p^{3/2} \) of the BD/RD model [2]. For small \( p \) values \( \lambda \) has a power law dependence with \( p \), while for \( p \rightarrow 1 \), \( \lambda(p) \rightarrow 0 \) with a fast decay. Now we proceed to generalize Eq. (1) for the CGM model applying the following transformation: \( h' = h \, f(p) \, b^\alpha, \, x' = b \, x \) and \( t' = g(p) \, t \, b^\gamma \). where \( b \) is the length transformation. As the interface evolution of these CGM are independent of \( b \) they can be described, after applying the transformations defined above, by the generalized continuous equation:

\[
\frac{dh(x,t)}{dt} = \nu(p) \frac{\partial^2 h(x,t)}{\partial^2 x} + \lambda(p) \left( \frac{\partial h(x,t)}{\partial x} \right)^2 + D(p) \eta(x,t)
\]

with

\[
\nu(p) = \nu_0 \, g(p) \\
\lambda(p) = \lambda_0 \, f(p) \, g(p) \\
D(p) = \frac{g(p)}{f(p)^2}
\]

where \( g(p) \) is related to the characteristic time scale when the correlations begins to dominate the dynamic of the interface and \( f(p) \) is related to the saturation length scale. Notice that for the EW universality class \( \lambda_0 = 0 \) Using the exact results from Sect II,

\[
f(p) \sim \begin{cases} 
p^{1/2}, & \text{for BD/RD and CEW/RD when } p \rightarrow 0, \\
p, & \text{for CEW/RD when } p \rightarrow 1,
\end{cases}
\]
FIG. 4: Log-Log plot of \( W f(p) \) as function of \( t g(p) \) for \( L = 8192 \) and different \( p \) values. In (a) \( p = 0.02 (\circ), p = 0.04 (\square) \) and \( p = 0.08 (\triangle) \), \( g(p) = p \) and \( f(p) = p^{1/2} \). In (b) \( p = 0.32 (\circ), p = 0.64 (\square) \) and \( p = 1.0 (\diamond) \), \( g(p) = p^2 \) and \( f(p) = p \). Notice the departure from the EW scaling behavior for \( p = 0.32 \) and \( g(p) = p^2 \), for BD/RD and CEW/RD when \( p \to 0 \), \( \lambda(p) \to \left\{ \begin{array}{ll} \lambda_0 p^{3/2}, & \text{for BD/RD when } p \to 0, \\
0, & \text{for CEW/RD when } p \to 1, \end{array} \right. \)
and \( D(p) \sim D_0 \) for BD/RD and CEW/RD independent of \( p \) Thus, for the CGM belonging to the KPZ (EW) universality class the evolution of the interface is given by:

\[
\frac{dh}{dt} = \nu_0 \eta(t), \quad \text{for BD/RD and CEW/RD when } p \to 0, \\
\nu_0 p^2 \frac{\partial^2 h(x,t)}{\partial x^2} + \eta(t), \quad \text{for CEW/RD when } p \to 1,
\]

The scaling behavior of \( W[11] \) is then:

\[
W f(p)/L^\alpha \sim \left\{ \begin{array}{ll} F \left( g(p) \lambda_0 \sqrt{\frac{\nu_0}{\nu_0} \frac{t}{L^2}} \right), & \text{for KPZ}, \\
F \left( g(p) \nu_0 \frac{t}{L^2} \right), & \text{for EW}, \end{array} \right.
\]
that after replacing \( f(p) \) and \( g(p) \) given by Eq. (12) and Eq. (13) leads to the exact scaling of \( W \) predicted by Eq. (2) with the exact values of \( \delta \) and \( y \) derived in Sec II. In Fig. 4 we show the Log-Log plot of \( W f(p) \) as function of \( t g(p) \) in the two limiting \( p \) values for fixed \( L \). The results are in agreement with our exact results (See Sec II) and our scaling ansatz (Eq. (17)).

V. CONCLUSIONS

We derive analytically the \( p \) dependance in the scaling behavior in two CGMs named BD/RD and EW/RD. Exact scaling exponents are derived and are in agreement with previously conjectured values [1, 2]. To our knowledge these exact scaling behaviors were not analytically derived before. This derivation allows us to compute the scaling behaviors of the coefficients of the continuous equations that describe their universality classes. We introduce the CEW/RD model to show that not all CGMs can be represented by an unique universality class. The CEW/RD is an EW in the limit \( p \to 1 \) while in the other limit \( p \to 0 \) it follows the strong coupling behavior the KPZ equation. Our simulation results are in excellent agreement with the analytic predictions.

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